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Existence of Solutions to Models of Age-Dependent Populations with Finite Life Span

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The McKendrick–Von Foerster population balance equation is considered in the case that individuals have a finite maximum life span. This maximum age is allowed to depend on time and total population density. The Banach fixed point theorem is used to establish the existence and uniqueness of a solution for various birth and death dynamics. © 1986 Academic Press, Inc.

1. INTRODUCTION

Let $\rho = \rho(a, t)$ represent a population density of individuals of age a at time t. The McKendrick–Von Foerster population balance equation [1, 4] states

$$D\rho = -\lambda\rho \tag{1.1}$$

where

$$D\rho = \lim_{h \to 0} \frac{\rho(a+h, t+h) - \rho(a, t)}{h}$$

and λ, the death modulus, is a positive quantity representing the death rate of individuals of age a at time t. If the death modulus is dependent on population density, the governing equation is nonlinear in ρ. At points where ρ is continuously differentiable in a and t

$$D\rho = \rho_t + \rho_a$$

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where the subscripts denote partial differentiation. In addition to the balance Eq. (1.1), it is assumed that the population's birth dynamics provide a knowledge of $\rho(0, t)$ for $t > 0$ and that at $t = 0$, the age dependence in the population density is specified by $\rho(a, 0)$.

In most studies of Eq. (1.1) it is assumed that it is mathematically possible for some individuals to live to an arbitrarily advanced age. Our goal in this paper is to establish the existence of solutions to Eq. (1.1) in cases where individuals do not live beyond age $a = L$. The first case considered is $\rho(a, t) \equiv 0$, $a > L(t)$, where L is a known function of t. In this case two types of birth dynamics are considered, namely

$$\rho(0, t) = B(P(t)) \tag{1.2}$$

and

$$\rho(0, t) = \int_0^{L(t)} \beta(a, P(t))\, \rho(a, t)\, da \tag{1.3}$$

where

$$P(t) = \int_0^{L(t)} \rho(a, t)\, da$$

denotes the total population density at time t. Even though Eq. (1.2) is a special case of Eq. (1.3), separate existence proofs are presented. The reason for this is that the proofs are constructive and the proof and the numerical algorithm associated with Eq. (1.2) is considerably simpler than what is required for Eq. (1.3).

As motivation for allowing a maximum life span of the form $L = L(t)$ consider insects whose maximum life span is temperature controlled. If temperature is viewed as a function of time then the case $L = L(t)$ arises.

The second case considered is that in which the maximum life span is dependent on total population density, $L = L(P(t))$, where

$$P(t) = \int_0^{L(P(t))} \rho(a, t)\, da.$$

This case arises, for example, in populations where maximum life span is limited by the availability of resources. With a fixed resource base, L would be expected to decrease with increasing P.

Populations governed by Eq. (1.1) in which individuals have finite life span have been considered in [3]. There, the maximum life span L was assumed to be a constant. Also, except in a neighborhood of $a = L$, the death modulus λ was assumed to satisfy $0 \leqslant \lambda \leqslant 1$. The existence and uniqueness arguments in the next sections rely on many of the techniques developed in [2].

2. Problem 1.

Our concern in Sections 2–4 is the existence and uniqueness of a solution of the balance equation (1.1). We will consider three different problems according to different assumptions on the maximum life span, the birth process, and the death modulus. The first two problems involve a death modulus of a form which necessitates a finite life span. For a death modulus of the form

$$\lambda(t, a, P)=\frac{\lambda_0}{L(t, P(t))-a} \tag{2.1}$$

where $\lambda_0 \geqslant 0$ is a constant, it follows that as a approaches $L(t, P(t))$, the death rate approches infinity. As Theorem (2.3) shows, such a death modulus ensures that individuals do not live past age $a=L$. Consider the following system of equations:

$$D\rho+\frac{\lambda_0}{L(t)-a}\rho=0 \qquad 0<a<L(t), \qquad 0<t. \tag{2.2a}$$

$$\rho(a, 0)=\phi(a) \qquad 0\leqslant a. \tag{2.2b}$$

$$P(t)=\int_0^{L(t)} \rho(a, t)\, da \qquad 0\leqslant t. \tag{2.2c}$$

$$\rho(0, t)=B(P(t)) \qquad 0<t. \tag{2.2d}$$

We refer to Eqs. (2.2a)–(2.2d) as Problem 1. Our analysis of Problem 1 is based on the following assumptions:

(H1) $L(t)$ is positive, continuously differentiable and $L'(t)<1$ for $t\geqslant 0$.

(H2) $\phi(a)$ is nonnegative, continuous and $\phi(a)\equiv 0$ for $a\geqslant L(0)$.

(H3) $B(P)$ is nonnegative, continuous on $[0, \infty)$, $B(0)=0$, and B satisfies the Lipschitz condition

$$|B(P_1)-B(P_2)|\leqslant \alpha\,|P_1-P_2|, \qquad \alpha>0, \quad P_1, P_2\in[0, \infty).$$

Definition 2.1. We say ρ is a solution of Problem 1 on $[0, \infty)$ if

(i) $\rho\geqslant 0$, $D\rho$ exists, $\rho(\cdot, t)$ is integrable over the interval $[0, L(t)]$, $t\geqslant 0$ and ρ satisfies (2.2a)–(2.2d).

(ii) $P(t)$ defined by (2.2c) is continuous for $t\geqslant 0$.

To discuss the existence and uniqueness for a solution of Problem 1, first we show that this problem is equivalent to an integral equation for the total population $P(t)$. Then we show this integral equation has a unique solution.

Let ρ be the solution of Problem 1 on $[0, \infty)$. Choose $(a_0, t_0) \in [0, L(t)) \times [0, \infty)$ and consider the characteristic $a = a_0 + h$, $t = t_0 + h$. Let

$$\hat{\rho}(h) = \rho(a_0 + h, t_0 + h).$$

Then it follows from (2.2a) that

$$\frac{d\hat{\rho}}{dh} + \frac{\lambda_0}{L(t_0 + h) - (a_0 + h)}\, \hat{\rho} = 0. \tag{2.3}$$

Equation (2.3) has the unique solution

$$\rho(a_0 + h, t_0 + h) = \rho(a_0, t_0) \exp\left[-\int_0^h \frac{\lambda_0}{L(t_0 + \tau) - (a_0 + \tau)}\, d\tau\right] \tag{2.4}$$

for $h < L - a_0$.

Setting $a_0 = a - t$, $t_0 = 0$, and $h = t$ in Eq. (2.4) and using (2.2b), we obtain

$$\rho(a, t) = \phi(a - t) \exp\left[-\int_0^t \frac{\lambda_0}{L(\tau) - (\tau + a - t)}\, d\tau\right], \qquad 0 \leqslant t \leqslant a < L, \tag{2.5}$$

and setting $a_0 = 0$, $t_0 = t - a$, $h = a$, and using (2.2d), we find

$$\rho(a, t) = B(P(t - a)) \exp\left[-\int_0^a \frac{\lambda_0}{L(t - a + \tau) - \tau}\, d\tau\right], \qquad 0 \leqslant a < t \leqslant L. \tag{2.6}$$

Substituting (2.5) and (2.6) in (2.2c) yields

$$P(t) = G(t) + \int_0^t K(t, a)\, B(P(a))\, da, \qquad 0 \leqslant t \leqslant T, \tag{2.7}$$

where $G(t)$ and $K(t, a)$ are continuous and defined by

$$G(t) = \int_0^{L(t) - t} \phi(a) \exp\left[-\int_0^t \frac{\lambda_0}{L(\tau) - (\tau + a)}\, d\tau\right] da, \qquad 0 \leqslant t \leqslant T, \tag{2.8}$$

$$K(t, a) = \exp\left[-\int_a^t \frac{\lambda_0}{L(\tau) - (\tau - a)}\, d\tau\right], \qquad 0 \leqslant a \leqslant t \leqslant T, \tag{2.9}$$

and T in Eqs. (2.7)–(2.9) is the solution of $L(t) - t = 0$. Note that by (H1), $L(t) - t \geqslant 0$ for $0 \leqslant t \leqslant T$, and hence $G(t)$ defined by (2.8) is nonnegative. If we assume $B(P(0)) = \phi(0)$, then ρ defined by (2.5) and (2.6) is continuous along the characteristic $a = t$.

THEOREM 2.1. *If ρ is a solution of Problem 1 on $[0, \infty)$, then $P(t)$ satisfies the nonlinear integral equation (2.7). If $P(t)$ is a nonnegative continuous function satisfying (2.7) and ρ is defined on $[0, L) \times [0, \infty)$ by (2.5) and (2.6), then ρ is a solution of Problem 1 on $[0, T]$.*

Proof. We have already established the first part of the theorem. For the second part, it is clear from (2.5) and (2.6) that $\rho \geqslant 0$, $\rho(\cdot, t)$ is integrable over $[0, L]$, $\rho(a, 0) = \phi(a)$, $\rho(0, t) = B(P(t))$. It remains to show that $D\rho$ exists and (2.2a) holds. Recall that

$$D\rho = \lim_{h \to 0} \frac{\rho(a+h, t+h) - \rho(a, t)}{h}.$$

Using (2.4), we have

$$D\rho = \rho(a, t) \lim_{h \to 0} \frac{\exp[-\int_0^h \lambda_0/(L(t+\tau) - (a+\tau))\, d\tau] - 1}{h} = \frac{-\lambda_0}{L(t) - a} \rho(a, t).$$

So the proof of Theorem 2.1 is completed.

THEOREM 2.2. *Problem 1 has a unique solution on $[0, T]$ where T is the solution of $L(t) - t = 0$.*

Proof. In view of Theorem 2.1, it suffices to show that the integral equation (2.7) has a unique solution for $P(t)$ which is positive and continuous on $[0, T]$.

Define a sequence of approximations by

$$P_0(t) = G(t),$$

and for $n \geqslant 1$,

$$P_n(t) = G(t) + \int_0^t K(t, a)\, B(P_{n-1}(a))\, da, \qquad 0 \leqslant t \leqslant T. \tag{2.10}$$

It is clear that all $P_n(t)$ are continuous and positive on $[0, T]$. We will show that the sequence $\{P_n(t)\}$ converges uniformly. Consider the series

$$\sum_{n=1}^{\infty} [P_n(t) - P_{n-1}(t)];$$

we have

$$|P_1(t) - P_0(t)| = \left| \int_0^t K(t, a)\, B(G(a))\, da \right| \leqslant \int_0^t B(G(a))\, da$$

and

$$B(G(t)) = |B(G(t)) - B(0)| \leqslant \alpha |G(t)| \leqslant \alpha M$$

where $M = \sup_{0 \leqslant t \leqslant T} G(t)$. Therefore,

$$|P_1(t) - P_0(t)| \leqslant \alpha M t. \tag{2.11}$$

For $n \geqslant 2$, we have from (2.10)

$$|P_n(t) - P_{n-1}(t)| \leqslant \alpha \int_0^t |P_{n-1}(a) - P_{n-2}(a)| \, da, \tag{2.12}$$

and hence from (2.11) and (2.12), we have for $n = 2$,

$$|P_2(t) - P_1(t)| \leqslant M(\alpha t)^2/2!, \qquad 0 \leqslant t \leqslant T, \tag{2.13}$$

and by induction

$$|P_n(t) - P_{n-1}(t)| \leqslant M(\alpha t)^n/n! \leqslant M(\alpha T)^n/n!, \qquad n \geqslant 1.$$

Since $\sum_{n=0}^{\infty} (\alpha T)^n/n! = e^{\alpha T}$, it follows by the Weierstrass M-test that $\sum_{n=1}^{\infty} [P_n(t) - P_{n-1}(t)]$ converges uniformly and therefore $\{P_n(t)\}$ converges uniformly. Let $P(t) = \lim_{n \to \infty} P_n(t)$, then $P(t)$ is nonnegative and continuous on $[0, T]$ and satisfies (2.7). To show that $P(t)$ is unique, assume $Q(t)$ is continuous and satisfies

$$Q(t) = G(t) + \int_0^t K(t, a) \, B(Q(a)) \, da, \qquad 0 \leqslant t \leqslant T. \tag{2.14}$$

From (2.10) and (2.14), one obtains

$$\begin{aligned} |P_1(t) - Q(t)| &\leqslant \int_0^t |B(G(a)) - B(Q(a))| \, da \\ &\leqslant \alpha \int_0^t |G(a) - Q(a)| \, da. \end{aligned}$$

By (2.14), we have

$$|G(t) - Q(t)| \leqslant \alpha N t$$

where $N = \sup_{0 \leqslant t \leqslant T} Q(t)$. Hence,

$$|P_1(t) - Q(t)| \leqslant N(\alpha t)^2/2!$$

and by induction, for $n \geqslant 1$,

$$|P_n(t) - Q(t)| \leqslant N(\alpha t)^{n+1}/(n+1)! \leqslant N(\alpha T)^{n+1}/(n+1)!.$$

This shows that $P_n(t) \to Q(t)$ uniformly and therefore $P(t) \equiv Q(t)$ for $0 \leqslant t \leqslant T$. This completes the proof of Theorem 2.2.

It has been shown that Problem 1 has a unique solution on $[0, T]$, where T is the solution of $L(t) - t = 0$. To show that Problem 1 has a unique solution for all time t, we translate the ta-coordinate system so that the origin of the new system has coordinates in $(T, 0)$ in ta-system. Let T_1 be the solution of $L(t) - t = 0$ with respect to the new system. By Theorem 2.2, Problem 1 has a unique solution on $[0, T]$. Continuing in this way, one can show that Problem 1 has a unique solution for all time t, provided $L(t)$ is positive and bounded away from zero for all t.

THEOREM 2.3. *If L is positive and continuously differentiable for $t \geqslant 0$ and $L'(t) < 1$, then*

$$\lim_{\substack{a \to L \\ a - t = c}} \rho(a, t) = 0$$

where c is a constant.

Proof. Let $t_0 > 0$ be arbitrary and let $L_0 = L(t_0)$. Consider the characteristic $t = t_0 + h$, $a = L_0 + h$, and set

$$u(h) = \rho(L_0 + h, t_0 + h)$$

for $h_1 < h < 0$. From Eq. (2.2a), we obtain

$$\frac{du}{dh} + \frac{\lambda_0 u}{L(t_0 + h) - (L_0 + h)}. \tag{2.15}$$

According to Taylor's theorem,

$$L(t_0 + h) = L_0 + hL'(t_0 + \hat{h}) \tag{2.16}$$

where $h_1 \leqslant h < \hat{h} \leqslant 0$. Equations (2.15) and (2.16) imply that

$$\frac{du}{dh} \leqslant \frac{\lambda_0 u}{(1 - L^*)\, h}, \qquad h_1 < h < 0$$

where $L^* < 1$ is the maximum of $L'(t_0 + \hat{h})$ for $h_1 \leqslant h \leqslant 0$. The above differential inequality shows that

$$|u(h)| \leqslant \text{CONST. } |h|^{\lambda_0/1 - L^*}$$

and therefore $u(h) \to 0$ as $h \to 0^-$, which is sufficient to establish the theorem.

The above theorem shows that, with a death modulus as in (2.2a), no individuals live past age $a = L$.

3. Problem 2

Our concern in this section is the solution of the following system of equations:

$$D\rho + \frac{\lambda_0}{L(t) - a}\rho = 0, \qquad 0 \leqslant a < L(t), 0 < t \tag{3.1a}$$

$$\rho(a, 0) = \phi(a), \qquad 0 \leqslant a \tag{3.1b}$$

$$P(t) = \int_0^{L(t)} \rho(a, t)\, da, \qquad 0 \leqslant t \tag{3.1c}$$

$$\rho(0, t) = \int_0^{L(t)} \beta(a, P(t))\, \rho(a, t)\, da, \qquad 0 < t. \tag{3.1d}$$

We refer to Eqs. (3.1a)–(3.2d) as Problem 2. Note that in this problem $\rho(0, t)$, the birth process, is given in terms of $\beta(a, P(t))$, the birth-modulus. To analyze Problem 2 we make the following assumptions:

(H1) $L(t)$ is positive, continuously differentiable for $t \geqslant 0$ and $L'(t) < 1$.

(H2) $\phi(a)$ is nonnegative, continuous and $\phi(a) \equiv 0$ for $a \geqslant L(0)$.

(H3) $\beta(a, P)$ is nonnegative and continuous on $[0, \infty) \times [0, \infty)$, $\beta_P(a, P)$ exists for all $a \geqslant 0$ and $P \geqslant 0$. Also, $\beta(\cdot, P)$ and $\beta_P(\cdot, P)$, as functions of P, belong to $C(R^+ : L_\infty(R^+))$, the set of all continuous functions from R^+ to $L_\infty(R^+)$.

Here again we say ρ is a solution of Problem 2 on $[0, \infty)$ if

(i) $\rho \geqslant 0$, $D\rho$ exists, $\rho(\cdot, t)$ is integrable over the interval $[0, L(t)]$, $0 \leqslant t$ and ρ satisfies (3.1a)–(3.1d).

(ii) $P(t)$ defined by (3.1c) is continuous for $t \geqslant 0$.

Let ρ be a solution of Problem 2 on $[0, \infty)$, then by an argument similar to Problem 1, one can show that

$$\rho(a, t) = \phi(a - t) \exp\left[-\int_0^t \frac{\lambda_0}{L(\tau) - (a - t + \tau)}\, d\tau \right], \quad 0 \leqslant t \leqslant a < L(t) \tag{3.2}$$

$$\rho(a, t) = B(t - a) \exp\left[-\int_0^a \frac{\lambda_0}{L(t - a + \tau) - \tau}\, d\tau \right], \quad 0 \leqslant a < t \tag{3.3}$$

where

$$B(t) = \rho(0, t), \qquad 0 < t.$$

If we assume $B(0) = \phi(0)$, then $\rho(a, 0)$ defined by (3.2) and (3.3) is continuous along $t = a$. Also, $B(t)$ will be continuous for $t \geqslant 0$ if ϕ satisfies the following condition:

$$\phi(0) = \int_0^{L(0)} \beta(a, P(0))\, \phi(a)\, da.$$

Substituting (3.2) and (3.3) into (3.1c) and (3.1d), one obtains

$$P(t) = \int_0^t M(a, t, L)\, B(a)\, da + \int_0^{L(t)-t} N(a, t, L)\, \phi(a)\, da, \qquad 0 \leqslant t \leqslant T, \tag{3.4}$$

$$\begin{aligned} B(t) = &\int_0^t \beta(t-a, P(t))\, M(a, t, L)\, B(a)\, da \\ &+ \int_0^{L(t)-t} \beta(a+t, P(t))\, N(a, t, L)\, \phi(a)\, da, \qquad 0 \leqslant t \leqslant T \end{aligned} \tag{3.5}$$

where

$$M(a, t, L) = \exp\left[-\int_a^t \frac{\lambda_0}{L(\tau) - (\tau - a)}\, d\tau\right], \qquad 0 \leqslant a \leqslant t \leqslant T \tag{3.6}$$

$$N(a, t, L) = \exp\left[-\int_0^t \frac{\lambda_0}{L(\tau) - (\tau + a)}\, da\right], \qquad 0 \leqslant t \leqslant a < L(t) - t, \qquad t \leqslant T. \tag{3.7}$$

In Eqs. (3.1) through (3.7), T is restricted to be any positive number with $T \leqslant T^*$, where T^* is the smallest positive root of the equation $L(t) - t = 0$. By (H1), $L(t) - t \geqslant 0$ for $0 \leqslant t \leqslant T$.

THEOREM 3.1. *If ρ is a solution of Problem* 2 *on* $[0, \infty)$ *then $P(t)$ and $B(t)$ satisfy the integral equations* (3.4) *and* (3.5). *Conversely, if $P(t)$ and $B(t)$ are nonnegative continuous functions satisfying* (3.4) *and* (3.5) *and if ρ is defined on* $[0, L(t)) \times [0, \infty)$ *by* (3.2) *and* (3.3), *then ρ is a solution of Problem* 2 *on* $[0, T]$.

The proof of this theorem is virtually identical to that of Theorem 2.1 and thus will be omitted.

THEOREM 3.2. *There exists a $T>0$ with $T \leqslant T^*$ such that Problem 2 has a unique solution on* $[0, T]$.

Proof. Let

$$C^+[0, T] = \{f : f \geqslant 0, f \in C[0, T]\}.$$

In view of Theorem 3.1, it suffices to show that the pair of integral equations (3.4) and (3.5) has a unique solution $(P(t), B(t))$ with $P, B \in C^+[0, T]$.

For fixed $P \in C^+[0, T]$, the linear integral equation (3.5) can be solved uniquely for B. This can be shown by the method used in the proof of Theorem 2.2. Define this solution by

$$B(t) = \mathscr{B}_T(P)(t), \qquad 0 \leqslant t \leqslant T. \tag{3.8}$$

We substitute (3.8) into (3.4) and define an operator

$$\mathscr{A}_T : C^+[0, T] \to C^+[0, T]$$

by

$$\mathscr{A}_T(P)(t) = \int_0^t M(a, t, L)\, \mathscr{B}_T(P)(a)\, da + \int_0^{L(t)-t} N(a, t, L)\, \phi(a)\, da. \tag{3.9}$$

Now we show that there exists a $T>0$ such that the operator $\mathscr{A}_T$ has a unique fixed point which is sufficient to establish the theorem.

Consider the Banach space $C[0, T]$ with norm given by

$$\|P\|_T = \sup\{P(t) : 0 \leqslant t \leqslant T\}.$$

Choose $r>0$ and set

$$S(T) = \{P \in C^+[0, T] : \|P - P_0\|_T \leqslant r\},$$

where

$$P_0 = P(0) = \int_0^{L(0)} \phi(a)\, da.$$

It is clear that $S(T)$ is a closed subset of $C[0, T]$. We show that $\mathscr{A}_T$ maps $S(T)$ into itself and is a contraction on $S(T)$ which by the contraction mapping theorem implies that $\mathscr{A}_T$ has a unique fixed point. Let

$$\beta_0 = \sup\{\beta(a, P) : P \in S(T), a \geqslant 0\}, \tag{3.10}$$

$$\beta_1 = \sup\{\beta_P(a, P) : P \in S(T), a \geqslant 0\}. \tag{3.11}$$

Let $P \in S(T)$. Using Eqs. (3.5) through (3.11), we have

$$\mathscr{B}_T(P)(t) \leqslant \beta_0 \int_0^t \mathscr{B}_T(P)(a)\, da + \beta_0 \int_0^{L(0)} \phi(a)\, da$$

$$\leqslant \beta_0 \int_0^t \mathscr{B}_T(P)(a)\, da + \beta_0 P_0 \tag{3.12}$$

and by Gronwall's inequality, Eq. (3.12) implies that

$$\mathscr{B}_T(P)(t) \leqslant \beta_0 P_0 e^{\beta_0 t}. \tag{3.13}$$

Now using Eqs. (3.9) and (3.13), we can write

$$|\mathscr{A}_T(P)(t) - P_0| \leqslant \left| \int_0^t M(a, t, L)\, \mathscr{B}_T(P)(a)\, da \right| + |\psi(t)|$$

$$\leqslant \int_0^t \beta_0 P_0 e^{\beta_0 a}\, da + |\psi(t)| \leqslant P_0 T e^{\beta_0 T} + |\psi(t)|,$$

where

$$\psi(t) = \int_0^{L(t)-t} N(a, t, L)\, \phi(a)\, da - \int_0^{L(0)} \phi(a)\, da.$$

Since $\lim_{t \to 0^+} \psi(t) = 0$, it follows that, given $r > 0$, there exists a $T' > 0$ such that

$$|\psi(t)| < r/2 \qquad \text{for} \quad t < T'$$

and hence for $0 \leqslant t \leqslant T < T'$, we obtain

$$|\mathscr{A}_T(P)(t) - P_0| \leqslant P_0 T e^{\beta_0 T} + r/2.$$

Therefore, for T sufficiently small

$$|\mathscr{A}_T(P)(t) - P_0| \leqslant r,$$

or

$$\|\mathscr{A}_T(P) - P_0\|_T \leqslant r$$

and this proves $\mathscr{A}_T(P) \in S(T)$.

Next, let $P_1, P_2 \in S(T)$. Employing Eq. (3.5), we have

$$|\mathscr{A}_T(P_1)(t) - \mathscr{A}_T(P_2)(t)| \leqslant \int_0^t |\mathscr{B}_T(P_1)(a) - \mathscr{B}_T(P_2)(a)|\, da. \tag{3.14}$$

Let

$$f(t) = \mathscr{B}_T(P_1)(t) - \mathscr{B}_T(P_2)(t), \tag{3.15}$$

then by Eq. (3.5), we find

$$f(t)=\int_0^t \beta(t-a, P_2(t)(\mathscr{B}_T(P_1)(a)-\mathscr{B}_T(P_2)(a))\, M(a, t, L)\, da + g(t), \tag{3.16}$$

where

$$\begin{aligned} g(t)=&\int_0^t [\beta(t-a, P_1(t))-\beta(t-a, P_2(t))]\, M(a, t, L)\, \mathscr{B}_T(P_1)(a)\, da \\ &+\int_0^t [\beta(a+t, P_1(t))-\beta(a+t, P_2(t))]\, N(a, t, L)\, \phi(a)\, da. \end{aligned} \tag{3.17}$$

From Eq. (3.16) one can write

$$f(t)\leqslant \beta_0 \int_0^t f(a)\, da + |g(t)|,$$

and by Gronwall's inequality

$$f(t)\leqslant |g(t)| + \beta_0 \int_0^t |g(a)|\, e^{\beta_0(t-a)}\, da, \qquad 0\leqslant t\leqslant T. \tag{3.18}$$

Using the mean value theorem in Eq. (3.17) and employing Eqs. (3.11) and (3.13), we find

$$|g(t)|\leqslant \beta_1 P_0 e^{\beta_0 t}\, \|P_1-P_2\|_T. \tag{3.19}$$

Now Eqs. (3.18) and (3.19) imply that

$$f(t)\leqslant \beta_1 P_0 e^{\beta_0 t}(1+\beta_0 t)\, \|P_1-P_2\|_T. \tag{3.20}$$

Finally, it follows from Eqs. (3.14), (3.15), and (3.20) and an integration by parts that

$$|\mathscr{A}_T(P_1)(t)-\mathscr{A}(P_2)(t)|\leqslant \beta_1 P_0\, \|P_1-P_2\|_T\, Te^{\beta_0 T}. \tag{3.21}$$

Let $k=\beta_1 P_0\|P_1-P_2\|_T e^{\beta_0 T}$, then for T sufficiently small, $0<k<1$, and therefore Eq. (3.21) implies that

$$\|\mathscr{A}_T(P_1)-\mathscr{A}_T(P_2)\|_T\leqslant k\, \|P_1-P_2\|_T.$$

This shows that $\mathscr{A}_T$ is a contraction and the proof of Theorem 3.2 is completed.

Theorem 2.5 is a local result. To obtain a global result, we need an a priori estimate for the local population P. To find such an estimate, we assume $\beta(a, P)$ is uniformly bounded, i.e.,

$$\beta^* = \sup_{\substack{a \geqslant 0 \\ P \geqslant 0}} \beta(a, P) < \infty.$$

Then from Eq. (3.5), we obtain

$$B(t) \leqslant \beta^* \int_0^t B(a)\, da + \beta^* \int_0^{L(t)-t} \phi(a)\, da$$

$$\leqslant \beta^* \int_0^t B(a)\, da + \beta^* P_0,$$

which implies that

$$B(t) \leqslant \beta^* P_0 e^{\beta^* t}.$$

Now using Eq. (3.4), we find

$$0 \leqslant P(t) \leqslant P_0 e^{\beta^* t}, \qquad 0 \leqslant t \leqslant T \tag{3.22}$$

which is an estimate for P.

Let $\bar{T} > 0$ be arbitrary. We show a solution for $P(t)$ exists on $[0, \bar{T}]$ which is sufficient to establish the existence of a solution for all t. Note that by the above remark, if a solution for $P(t)$ exists on $[0, \bar{T}]$, then $0 \leqslant P(t) \leqslant P_0 e^{\beta^* \bar{T}}$. In the proof of local existence, T, the time interval, is a continuous function of initial total population. Denote by $T(x)$ the time interval corresponding to initial population x. By the local result, we know that the solution exists on $[0, T^*]$, where

$$T^* = \min_{0 \leqslant x \leqslant P_0 e^{\beta^* T}} T(x).$$

Let P_1 be the total population at time $t = T^*$, then clearly $0 \leqslant P_1 \leqslant P_0 e^{\beta^* T}$. Taking P_1 as the initial total population, we know that the solution exists on $[T^*, 2T^*]$. Continuing this way, after a finite number of steps, we conclude that the solution exists on $[0, \bar{T}]$.

4. Problem 3

In this section we discuss the existence and uniqueness of a solution of the following equations:

$$D\rho + \lambda(a, P(t))\rho = 0, \qquad 0 \leqslant a \leqslant L(P),\ 0 < t \leqslant T, \tag{4.1a}$$

$$\rho(a, 0) = \phi(a), \qquad 0 \leqslant a \tag{4.1b}$$

$$P(t) = \int_0^{L(P(t))} \rho(a, t)\, da, \qquad 0 \leqslant t \leqslant T \tag{4.1c}$$

$$\rho(0, t) = B(P(t)), \qquad 0 < t \leqslant T. \tag{4.1d}$$

We refer to Eqs. (4.1a)–(4.1d) as Problem 3. In our study of Problem 3, we use the following assumptions:

(H1) $L(P)$ is a positive continuous function of P on $[0, \infty)$ and satisfies the Lipschitz condition

$$|L(P_1) - L(P_2)| \leqslant \alpha |P_1 - P_2|,\ P_1, P_2 \in [a, \infty), \qquad \alpha > 0.$$

(H2) $\phi(a)$ is nonnegative continuous function and $\phi(a) \equiv 0$ for $a \geqslant L(P(0))$.

(H3) $B(P)$ is nonnegative continuous function of P on $[0, \infty)$ and satisfies

$$|B(P_1) - B(P_2)| \leqslant \beta |P_1 - P_2|,\ P_1, P_2 \in [0, \infty), \qquad \beta > 0.$$

(H4) $b = \sup_{P \geqslant 0} B(P)$ is finite.

(H5) $\lambda(a, P)$ is continuous on $[0, \infty) \times [0, \infty)$ and $\lambda_P(a, P)$ exists for all $a \geqslant 0$, $P \geqslant 0$ and $\lambda_P(\cdot, P)$, as a function of P, belong to $C(R^+ : L_\infty(R^+))$. Note that by (H2), the total population at $t = 0$ is given since

$$P(0) = \int_0^{L(P(0))} \rho(a, 0)\, da = \int_0^{\infty} \phi(a)\, da$$

and thus $L(P(0))$, the maximum life span at $t = 0$ is known.

DEFINITION 4.1. ρ is said to be a solution of Problem 3 on $[0, T]$ if

(i) $\rho \geqslant 0$, $D\rho$ exists and $\rho(\cdot, t)$ is integrable over the interval $[0, L(P(t))]$, $0 \leqslant t \leqslant T$ and ρ satisfies (4.1a)–(4.1d).

(ii) $P(t)$ defined by (4.1c) is continuous on $[0, T]$.

Let ρ be a solution of Problem 3 on $[0, T]$. Then using the same approach as we previously employed for Problems 1 and 2, we obtain the integral equation formulation

$$\rho(a, t) = \phi(a - t) \exp\left[-\int_0^t \lambda(a - t + \tau, P(\tau))\, d\tau \right], \tag{4.2}$$

for $0 \leqslant t \leqslant a \leqslant L(P)$, $0 \leqslant t \leqslant T$ and

$$\rho(a, t) = B(P(t-a)) \exp\left[-\int_0^a \lambda(\tau, P(t-a+\tau))\, d\tau\right] \tag{4.3}$$

for $0 \leqslant a < t$, $a \leqslant L(P)$, $0 \leqslant t \leqslant T$.

If we set $\rho(a, t) = 0$ for $a > L(P)$, then $\rho(a, t)$ is defined for all $a \geqslant 0$. If we assume $B(P(0)) = \phi(0)$, then $\rho(a, t)$ defined on $[0, L(P(t))] \times [0, T]$ by Eqs. (4.2) and (4.3) will be continuous along the line $t = a$.

Substituting Eqs. (4.2) and (4.3) into (4.1c), we obtain

$$P(t) = \int_0^t M(a, t, P)\, B(P(a))\, da + \int_0^{L(P)-t} N(a, t, P)\, \phi(a)\, da, \qquad 0 \leqslant t \leqslant T \tag{4.4}$$

where

$$M(a, t, P) = \exp\left[-\int_a^t \lambda(\tau - a, P(\tau))\, d\tau\right], \qquad 0 \leqslant a \leqslant t,\ a \leqslant L(P)$$

and

$$N(a, t, P) = \exp\left[-\int_0^t \lambda(\tau - a, P(\tau))\, d\tau\right], \qquad 0 \leqslant t \leqslant a \leqslant L(P). \tag{4.5}$$

Note that for T sufficiently small, $L(P(t)) - t > 0$ and hence $P(t)$ defined by (4.4) is nonnegative.

THEOREM 4.2. *If ρ is a solution of Problem 3 on $[0, T]$, then $P(t)$ satisfies the integral Eq. (4.4). If $P(t)$ is nonnegative continuous on $[0, T]$ satisfies (4.4) and ρ is defined on $[0, L(P(t))] \times [0, T]$ by (4.2) and (4.3), then ρ is a solution of Problem 3.*

Proof. See the proof of Theorem 2.1.

THEOREM 4.3. *There exists a $T > 0$ such that Problem 3 has a unique solution on $[0, T]$.*

Proof. In view of Theorem 4.2, it suffices to show that the integral equation (4.4) has a unique solution for $P(t)$ which is nonnegative and continuous on $[0, T]$.

Consider the Banach space $C[0, T]$ with norm given by

$$\|P\|_T = \sup_{0 \leqslant t \leqslant T} |P(t)| \qquad \text{for} \quad P \in C[0, T].$$

Choose $r > 0$ and let

$$S(T) = \{P \in C[0, T], P \geqslant 0, \|P - P_0\|_T \leqslant r\} \tag{4.6}$$

where $P_0 = P(0)$.

It is clear that $S(T)$ is a closed subset of $C[0, T]$. Define an operator $\mathscr{A}_T$: $S(T) \to C[0, T]$ by

$$\mathscr{A}_T(P)(t) = \int_0^t M(a, t, P)\, B(P(a))\, da + \int_0^{L(P(t)) - t} N(a, t, P)\, \phi(a)\, da. \tag{4.7}$$

As before, for T sufficiently small, $\mathscr{A}_T(P)(t)$ is nonnegative continuous function on $[0, T]$.

We use the contraction mapping theorem to show that the operator $\mathscr{A}_T$ has a unique fixed point which completes the proof of Theorem 4.2. Let $P \in S(T)$, then by (H4) and (4.5), one obtains

$$\begin{aligned}|\mathscr{A}_T(P(t)) - P_0| &= \left|\int_0^t M(a, t, P)\, B(P(a))\, da\right| \\ &\quad + \left|\int_0^{L(P) - t} N(a, t, P)\, \phi(a)\, da - \int_0^{L(P_0)} \phi(a)\, da\right| \\ &< bT + |F(t)|,\end{aligned}$$

where

$$F(t) = \int_0^{L(P) - t} N(a, t, P)\, \phi(a)\, da - \int_0^{L(P_0)} \phi(a)\, da.$$

Note that $F(t)$ is continuous on $[0, T]$ and $F(t) \to 0$ as $t \to 0^+$. Therefore, there exists $T' > 0$ such that

$$|F(t)| < r/2 \qquad \text{for} \quad t < T'$$

and hence, for $0 \leqslant t \leqslant T < T'$, we have

$$|\mathscr{A}_T(P)(t) - P_0| \leqslant bT + r/2 \leqslant r,$$

for T sufficiently small. Thus,

$$\|\mathscr{A}_T(P) - P_0\|_T \leqslant r,$$

and $\mathscr{A}_T$ maps $S(T)$ into itself. Now let $P_1, P_2 \in S(T)$, then

$$
\begin{aligned}
&|\mathscr{A}_T(P_1)(t) - \mathscr{A}_T(P_2)(t)| \\
&\quad = \int_0^t M_1 B_1 \, da - \int_0^t M_2 B_2 \, da + \int_0^{L_1 - t} N_1 \phi(a) \, da - \int_0^{L_2 - t} N_2 \phi(a) \, da \\
&\quad \leqslant \left| \int_0^t (M_1 - M_2) \, B_1 \, da \right| + \left| \int_0^t (B_1 - B_2) \, M_2 \, da \right| \\
&\qquad + \left| \int_0^{L_1 - t} N_1 \phi(a) \, da - \int_0^{L_2 - t} N_2 \phi(a) \, da \right| \\
&\quad \leqslant b \int_0^t |M_1 - M_2| \, da + \int_0^t |B_1 - B_2| \, da + |G(t)| \qquad (4.8)
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &= M(a, t, P_1), & M_2 &= M(a, t, P_2), \\
N_1 &= N(a, t, P_1), & N_2 &= N(a, t, P_2), \\
L_1 &= L(P_1(t), & L_2 &= L(P_2(t)), \\
B_1 &= B(P_1), & B_2 &= B(P_2),
\end{aligned}
$$

and

$$
G(t) = \int_0^{L_1 - t} N_1 \phi(a) - \int_0^{L_2 - t} N_2 \phi(a) \, da. \qquad (4.9)
$$

We have

$$
\begin{aligned}
|M_1 - M_2| &\leqslant |1 - M_2/M_1| \\
&= \left| 1 - \exp\left[\int_a^t (\lambda(\tau - a, P_1(\tau)) - \lambda(\tau - a, P_2(\tau))) \, d\tau \right] \right|.
\end{aligned}
$$

Using the inequality $|1 - e^x| \leqslant |x| \, e^{|x|}$ and the mean value theorem, we obtain

$$
\begin{aligned}
|M_1 - M_2| \leqslant & \left| \int_a^t (P_1(\tau) - P_2(\tau)) \, \lambda_P(\tau - a, \bar{P}(\tau)) \, d\tau \right| \\
& \times \exp\left[\int_a^t (P_1(\tau) - P_2(\tau)) \, \lambda_P(\tau - a, \bar{P}(\tau)) \, d\tau \right],
\end{aligned}
$$

where $P_2 < \bar{P} < P_1$. Let

$$
\lambda^* = \sup\{\lambda_P(a, P) \colon P \in S(T), \, a \geqslant 0\},
$$

then

$$\begin{aligned}|M_1 - M_2| &\leqslant \lambda^* |t-a| \sup_{0 \leqslant t \leqslant T} |P_1(t) - P_2(t)| \\ &\quad \times \exp[\lambda^* |t-a| \sup_{0 \leqslant t \leqslant T} |P_1(t) - P_2(t)|] \\ &\leqslant \lambda^* T \|P_1 - P_2\|_T \exp[\lambda^* T \|P_1 - P_2\|_T].\end{aligned}$$

Also,

$$\int_0^t |B_1 - B_2| \, da \leqslant \int_0^t \beta |P_1(a) - P_2(a)| \, da \leqslant \beta T \|P_1 - P_2\|_T.$$

Now we estimate the function $G(t)$ defined by (4.9). This function is continuous on $[0, T]$ and

$$\lim_{t \to 0^+} G(t) = \int_{L(P_1(0))}^{L(P_2(0))} \phi(a) \, da. \tag{4.10}$$

We have

$$\begin{aligned}\left| \int_{L(P_1(0))}^{L(P_2(0))} \phi(a) \, da \right| &\leqslant |L(P_1(0)) - L(P_2(0))| \, \phi(\hat{a}) \\ &\leqslant \alpha \, |P_1(0) - P_2(0)| \, \phi(\hat{a}) \\ &\leqslant \alpha \, \|P_1 - P_2\|_T \, \phi(\hat{a}),\end{aligned} \tag{4.11}$$

where

$$\min\{L(P_1(0)), L(P_2(0))\} < \hat{a} < \max\{L(P_1(0)), L(P_2(0))\}.$$

Therefore, it follows from Eqs. (4.10) and (4.11) that there exists a $T'' > 0$ such that

$$|G(t)| \leqslant \alpha \, \|P_1 - P_2\|_T \, \phi(\hat{a})$$

and hence for $0 \leqslant t \leqslant T < T''$, we obtain

$$|G(t)| \leqslant \alpha \, \|P_1 - P_2\|_T \, \phi(\hat{a}).$$

Now from Eq. (4.8), one can write

$$\begin{aligned}|\mathscr{A}_T(P_1)(t) - \mathscr{A}_T(P_2)(t)| &\leqslant b\lambda^* T^2 \|P_1 - P_2\|_T \exp[\lambda^* T \, \|P_1 - P_2\|_T] \\ &\quad + \beta T \|P_1 - P_2\|_T + \alpha \|P_1 - P_2\|_T \, \phi(\hat{a}) \\ &\leqslant (b\lambda^* T^2 \exp[\lambda^* T \|P_1 - P_2\|_T] \\ &\quad + \beta T + \alpha \phi(\hat{a})) \, \|P_1 - P_2\|_T.\end{aligned} \tag{4.12}$$

If r is chosen sufficiently small, then $\hat{a}$ is sufficiently close to $L(P(0))$, and hence $\phi(\hat{a})$ is sufficiently small. Therefore, if we set

$$K = b\lambda^* T^2 \exp[\lambda^* T \, \|P_1 - P_2\|_T] + \beta T + \alpha\phi(\hat{a}),$$

then for T and r sufficiently small, $0 < K < 1$, and it follows from Eq. (4.12) that

$$\|\mathscr{A}_T(P_1) - \mathscr{A}_T(P_2)\|_T \leqslant K \, \|P_1 - P_2\|_T.$$

This shows that $\mathscr{A}_T$ is a contraction on $S(T)$ and the proof of Theorem 4.3 is completed.

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